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On Unified Computational $\overline{H}$-function and $S_{V'}^{U'}$ polynomial

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Abstract

In this paper, I aim at evaluating unified infinite integral whose integrand involves the product of the $\overline{H}$-function and S_V^U polynomial. Further, its argument contains the factors of the form $x^{\lambda-1}(1+\alpha x^k)^{-\mu}(1+\beta x^l)^{-\nu}$. On account of the nature of the functions in the integrand, a large number of unknown integrals can be easily obtained from it by specializing the functions and parameters involved therein. For illustration, we establish three new integrals containing the functions such as: Fox H-function, Appell polynomial, Generalised Wright Hypergeometric function, Jacobi polynomial, Generalised Riemann Zeta function, Laguerre polynomial.

Keywords: Appell polynomial, S_V^U polynomial, Jacobi polynomial, Laguerre polynomial, $\overline{H}$-function, Fox H-function, Generalised Wright Hypergeometric function, Generalised Riemann Zeta function.

1 Introduction

1.1 $S_{V'}^{U'}$ Polynomial

The following polynomial is known as the $S_{V'}^{U'}$ Polynomial and was introduced by Srivastava [13], in 2006:

$$S_{V'}^{U'}[x] = \sum_{R'=0}^{[V'/U']} \frac{(-V')_{U'R'} A_{V',R'}}{R'!} x^{R'}, \qquad V' = 0, 1, 2, ... \tag{1}$$

U' represents a positive arbitrary integer, the $A_{V',R'}$ (coefficients) are always taken as constants, complex or real.
Taking U' = 1 in (1) $S_{V'}^{U'}$ polynomial reduces to Appell polynomial which was introduced by P. E. Appell [1], defined as follows:

$$A_{V'}(x) = \sum_{R'=0}^{V'} \frac{a_{\ V'-R'}}{R'!} x^{R'}, \qquad V' = 0, 1, 2... \tag{2}$$

$a_{V'-R'}$ are the complex coefficient and $a_0 \neq 0$.

1.2 $\overline{H}$-function

Inayat Hussain [6] introduces the $\overline{H}$-function, which was later studied by Bushman and Srivastava [2] and others. The utile representation of the $\overline{H}$-function in form of a series was obtained by Rathie [11], and is represented as follows (see, details, [8], [9]):

$$\overline{H}_{p',q'}^{m',n'} \left[z' \left| \begin{array}{cc} (e'_j, E'_j; \in_j)_{1,n'}, & (e'_j, E'_j)_{n'+1,p'} \\ \\ (f'_j, F'_j)_{1,m'}, & (f'_j, F'_j; \Im_j)_{m'+1,q'} \end{array} \right. \right] = \sum_{t'=0}^{\infty} \sum_{h'=1}^{m'} \overline{\Theta}(\mathfrak{s}_{t',h'}) z^{\mathfrak{s}_{t',h'}} \tag{3}$$

where,

$$\overline{\Theta}(\mathfrak{s}_{t',h'}) = \frac{\prod_{j=1,j\neq h'}^{m'} \Gamma(f'_j - F'_j \mathfrak{s}_{t',h'}) \prod_{j=1}^{n'} \left\{\Gamma(1 - e'_j + E'_j \mathfrak{s}_{t',h'})\right\}^{\in_j}}{\prod_{j=m'+1}^{q'} \left\{\Gamma(1 - f'_j + F'_j \mathfrak{s}_{t',h'})\right\}^{\Im_j} \prod_{j=n'+1}^{p'} \Gamma(e'_j - E'_j \mathfrak{s}_{t',h'})} \frac{(-1)^{t'}}{t'! F'_{h'}}, \tag{4}$$

$$\mathfrak{s}_{t',h'} = \frac{f'_{h'} + t}{F'_{h'}}$$

In continuation, we will also be using the following behaviour of the $\overline{H}^{m',n'}_{p',q'}[z']$ function for very large and small values of z given by Saxena et al. [12].

$$\overline{H}^{m',n'}_{p',q'}[z'] = O[|z'|^{\alpha}], \text{for small values of z', where} \quad \alpha = \min_{1\leq j\leq m'} \Re\left(\frac{f'_j}{F'_j}\right) \tag{5}$$

$$\overline{H}^{m',n'}_{p',q'}[z'] = O[|z'|^{\beta}], \text{for large values of z', where} \quad \beta = \max_{1\leq j\leq n'} \Re\left(\in_j \left(\frac{e_j - 1}{E'_j}\right)\right) \tag{6}$$

Considering $\in_i = \Im_j = 1 (i = 1, ..., n'; j = m' + 1, ..., q')$ in (3) $\overline{H}$-function will reduce to Fox H-function [14], which is defined and represented as follows:

$$H^{M',N'}_{P',Q'}[z'] = H^{M',N'}_{P',Q'}\left[z' \left| \begin{matrix} (a'_j, \alpha_j)_{1,P'} \\ \\ (b'_j, \beta_j)_{1,Q'} \end{matrix} \right.\right] \tag{7}$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{\mathfrak{L}} \Theta(\mathfrak{s}) z^{\mathfrak{s}} \, d\mathfrak{s} \tag{8}$$

$z' \in \mathbb{C} \setminus \{0\}$, $\mathrm{i} = \sqrt{-1}$, where $\mathbb{C}$ represents complex numbers,

$$\Theta(\mathfrak{s}) = \frac{\prod_{j=1}^{M'} \Gamma(b'_j - \beta'_j \mathfrak{s}) \prod_{j=1}^{N'} \Gamma(1 - a'_j + \alpha'_j \mathfrak{s})}{\prod_{j=M'+1}^{Q'} \Gamma(1 - b'_j + \beta'_j \mathfrak{s}) \prod_{j=N'+1}^{P'} \Gamma(a'_j - \alpha'_j \mathfrak{s})} \tag{9}$$

and

$$1 \leqq M' \leqq Q' \qquad \text{and} \qquad 0 \leqq N' \leqq P' \qquad (M', Q' \in \mathbb{N};\ N', P' \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}) \tag{10}$$

Here, product if empty is interpreted as 1 and $\mathfrak{L}$ is a Mellin-Barnes type contour in the complex $\mathfrak{s}$-plane with some indentations used to separate the two sets of the poles of given integrand $\Theta(\mathfrak{s})$ (see [7] and [14]).

2 Main Integrals

2.1 First Integral

$$\int_0^\infty x^{\lambda-1}(1+\alpha x^k)^{-\mu}(1+\beta x^l)^{-\nu} dx$$

$$= \frac{1}{k\Gamma(\mu)\Gamma(\nu)\alpha^{\frac{\lambda}{k}}} H_{2,2}^{2,2}\left[\frac{\beta}{\alpha^{\frac{l}{k}}} \,\middle|\, \begin{matrix} (1-\nu, 1), (1-\frac{\lambda}{k}, \frac{l}{k}) \\ \\ (0,1), (\mu - \frac{\lambda}{k}, \frac{l}{k}) \end{matrix}\right] \tag{11}$$

provided that the conditions given below are satisfied :-
(i) $l, k > 0$ (ii)$\Re(\lambda) > 0$
(iii) $\Re(k\mu + l\nu - \lambda) > 0$
Proof : To prove the First Integral (11). I express $(1+\beta x^l)^{-\nu}$ in the form of contour integral using [14] occurring in the left hand side. Now according to permissibility of the given condition I interchange the x-integral and contour integral.

Thus the L.H.S of (11) takes the form given below after a little simplification (say Δ)

$$\Delta = \frac{1}{2\pi \mathrm{i}\Gamma(\nu)k}\int_{\mathfrak{L}} \frac{1}{\alpha^{\frac{\lambda+l\mathfrak{s}}{k}}}\Gamma(\nu+\mathfrak{s})\Gamma(-\mathfrak{s})\beta^{\mathfrak{s}}d\mathfrak{s}\int_0^\infty t^{\frac{\lambda+l\mathfrak{s}}{k}-1}(1+t)^{-\mu}dt \tag{12}$$

Now I evaluate t-integral occurring in (12), finally reinterpreting the result obtained in terms of Fox H-function. I easily arrive at the R.H.S of (11) after a some simplification.

2.2 Second Integral

$$\int_0^\infty x^{\lambda-1}(1+\alpha x^k)^{-\mu}(1+\beta x^l)^{-\nu} S_{V'}^{U'}[z_2\, x^{\lambda_0}(1+\alpha x^k)^{-\mu_0}(1+\beta x^l)^{-\nu_0}]$$

$$\overline{H}_{p',q'}^{m',n'}\left[z_1 x^{\lambda_1}(1+\alpha x^k)^{-\mu_1}(1+\beta x^l)^{-\nu_1}\left| \begin{matrix} (e'_j, E'_j; \in_j)_{1,n'}, (e'_j, E'_j)_{n'+1,p'} \\ (f'_j, F'_j)_{1,m'}, (f'_j, F'_j; \Im_j)_{m'+1,q'} \end{matrix}\right.\right] dx$$

$$= \sum_{h=1}^{m}\sum_{t=0}^{\infty}\sum_{R'=0}^{[V'/U']} \frac{(-V')_{U'R'}A_{V',R'}}{R'!}\overline{\Theta}(\mathfrak{s}_{t,h}) z_1^{\mathfrak{s}_{t,h}} z_2^{R'} \frac{1}{k\Gamma(\mu+\mu_1\mathfrak{s}_{t,h}+\mu_0 R')}$$

$$\frac{1}{\alpha^{\frac{\lambda+\lambda_1\mathfrak{s}_{t,h}+\lambda_0 R'}{k}}} H_{2,2}^{2,2}\left[\frac{\beta}{\alpha^{\frac{l}{k}}}\left| \begin{matrix} (1-\nu-\nu_1\mathfrak{s}_{t',h'}-\nu_0 R', 1), (1-\frac{\lambda+\lambda_1\mathfrak{s}_{t',h'}+\lambda_0 R}{k}, \frac{l}{k}) \\ (0,1), (\mu+\mu_1\mathfrak{s}_{t',h'}+\mu_0 R-\frac{\lambda+\lambda_1\mathfrak{s}_{t',h'}+\lambda_0 R}{k}, \frac{l}{k}) \end{matrix}\right.\right] \tag{13}$$

provided the conditions given below are satisfied :-

(i) $\Re(k\mu + l\nu - \lambda + (k\mu_1 + l\nu_1 - \lambda_1)\max\limits_{1\le j\le n}\left(\in_j\left(\frac{e'_j-1}{E'_j}\right)\right)) > 0$

(ii) $\Re(\lambda + \lambda_1 \min\limits_{1\le j\le n'}\left(\frac{f'_j}{F'_j}\right)) > 0$ (iii) $l, k > 0$

(iv) min $\Re(\lambda,\, \lambda_1,\, \mu,\, \mu_1, \nu,\, \nu_1) \ge 0$ (not all are simultaneously zero)

Proof : To prove Integral (13), I express $S_{V'}^{U'}$ and $\overline{H}$-function in thir series form using (1) and (3) respectively. Further, I exchange the order of summation with x-integral (permissible under stated condition). Thus, the L.H.S of (13) takes the following form after some simplification (stated as Δ)

$$\Delta = \sum_{h'=1}^{m'} \sum_{t'=0}^{\infty} \sum_{R'=0}^{[V'/U']} \frac{(-V')_{U'R'} A_{V',R'}}{R'!} \overline{\Theta}(\mathfrak{s}_{t',h'})\, z_1^{\mathfrak{s}_{t',h'}}\, z_2^{R'} \int_0^{\infty} x^{\lambda+\lambda_1 \mathfrak{s}_{t',h'}+\lambda_0 R'-1}(1+\alpha x^k)^{-\mu-\mu_1 \mathfrak{s}_{t',h'}-\mu_0 R'} * (1+\beta x^l)^{-\nu-\nu_1 \mathfrak{s}_{t',h'}-\nu_0 R'} dx \tag{14}$$

Further, evaluating x-integral in (13) using (10), reinterpreting the obtained result in Fox H-function. After a little simplification, I can easily obtain R.H.S of (11).

2.3 Special Case

(i) In the Integral (13), if I take $\lambda_1 = \mu_1 = 0$ and also reduce $\overline{H}$-function to well established Fox H-function further reduce $S_{V'}^{U'}$ polynomial to Appell polynomial [1], I obtain the following integral.

$$\int_0^{\infty} x^{\lambda-1}(1+\alpha x^k)^{-\mu}(1+\beta x^l)^{-\nu} H_{p',q'}^{m',n'}\left[z_1(1+\beta x^l)^{-\nu_1} \left| \begin{matrix} (e'_j, E'_j)_{1,p'} \\ (f'_j, F'_j)_{1,q'} \end{matrix} \right. \right] A_{V'}[z_2 x^{\lambda_0}(1+\alpha x^k)^{-\mu_0}(1+\beta x^l)^{-\nu_0}]dx$$

$$= \sum_{R'=0}^{V'} \frac{a_{V'-R'}}{R'!} \frac{1}{k\Gamma(\mu+\mu_0 R')} \frac{1}{\alpha^{\frac{\lambda+\lambda_0 R'}{k}}} z_2^{R'}\, H_{1,0:p',q'+1;1,2}^{0,1:m',n';2,1} \left[\begin{matrix} z_1 \\ \\ \frac{\beta}{\alpha^{\frac{l}{k}}} \end{matrix} \left| \begin{matrix} A^* & : C^* \\ \\ B^* & : D^* \end{matrix} \right. \right] \tag{15}$$

where
$A^* = (1-\nu-\nu_0 R'; \nu_1, 1)$ $\quad B^* = -$
$C^* = (e'_j, E'_j)_{1,p'}; (1-\frac{\lambda+\lambda_0 R'}{k}, \frac{l}{k})$
$D^* = (f'_j, F'_j)_{1,q'}, (1-\nu-\nu_0 R', \nu_1); (0,1), (\mu+\mu_0 R' - \frac{\lambda+\lambda_0 R'}{k}, \frac{l}{k})$
provided that the conditions obtainable from (13) are satisfied.

(ii) Now, on taking $z_2 = 1$, $\mu_0 = \nu_0 = \mu_1 = \nu_1 = \lambda_1 = 0$ and reducing $\overline{H}$-function to Generalised Wright Hypergeometric function [4], [10] and reducing $S_{V'}^{U'}$ polynomial to Jacobi polynomial [15] I can easily obtain the following integral.

$$\int_0^\infty x^{\lambda-1}(1+\alpha x^k)^{-\mu}(1+\beta x^l)^{-\nu}\, {}_{p'}\overline{\psi}_{q'}\left[\begin{array}{c} (e'_j, E'_j; \in_j)_{1,p'} \\ \\ (f'_j, F'_j; \Im_j)_{1,q'} \end{array}; z_1\right] P_{V'}^{(\alpha_1,\beta_1)}(1-2x^{\lambda_0})dx$$

$$= \sum_{t'=0}^{\infty}\sum_{R'=0}^{V'} \frac{\prod_{j=1}^{p'}(\Gamma(e'_j+E'_j t))^{\in_j}}{\prod_{j=2}^{q'}(\Gamma(f'_j+F'_j t))^{\Im_j}} \frac{z_1^t}{t!} \frac{(-V')_{R'}(1+\alpha_1+\beta_1+V')_{R'}}{(1+\alpha_1)_{R'}\ R'!} x^{R'\lambda_0} \left(\begin{array}{c} V'+\alpha_2 \\ V' \end{array}\right)$$

$$\frac{1}{k\Gamma(\mu)\Gamma(\nu+\nu_1 t)} \frac{1}{\alpha^{\frac{\lambda+\lambda_0 R'}{k}}} H_{2,2}^{2,2}\left[\frac{\beta}{\alpha^{\frac{l}{k}}}\left|\begin{array}{c} (1-\nu-\nu_1 t, 1), (1-\frac{\lambda+\lambda_0 R'}{k}, \frac{l}{k}) \\ \\ (0,1), (\mu-\frac{\lambda+\lambda_0 R'}{k}, \frac{l}{k}) \end{array}\right.\right] \tag{16}$$

provided that the existing conditions of (13) are satisfied.

(iii) Again in the Integral (12), if I take $z_1 = 1$,$\mu_0 = \lambda_0 = \nu_1 = 0$ and reduce $\overline{H}$-function to well known Generalized Riemann Zeta function [3, 5] and reduce $S_{V'}^{U'}$ polynomial to well established Laguerre polynomial [14] by taking U' = 1, $A_{V',R'} = \left(\begin{array}{c} V'+\rho \\ V' \end{array}\right) \frac{1}{(\rho+1)_{R'}}$, I can easily obtain the following integral.

$$\int_0^\infty x^{\lambda-1}(1+\alpha x^k)^{-\mu}(1+\beta x^l)^{-\nu}\,\phi(x^{\lambda_1}(1+\alpha x^k)^{-\mu_1},\sigma,\eta)\,L_{V'}^{(\rho)}(z(1+\beta x^l)^{-\nu_0})dx$$

$$= \sum_{t=0}^{\infty}\sum_{R'=0}^{V'}\frac{1}{(\eta+t)^\sigma}\begin{pmatrix} V'+\rho \\ V' \end{pmatrix}\frac{z^{R'}}{R'!}\frac{(-V')_{R'}}{(\rho+1)_{R'}}\frac{1}{k\,\Gamma(\mu+\mu_1 t)\,\Gamma(\nu+\nu_0 R')}\frac{1}{\alpha^{\frac{\lambda+\lambda_1}{k}}}$$
$$H_{2,2}^{2,2}\left[\frac{\beta}{\alpha^{\frac{l}{k}}}\left|\begin{array}{c}(1-\nu-\nu_0 R',1),(1-\frac{\lambda+\lambda_1}{k},\frac{l}{k})\\ \\ (0,1),(\mu+\mu_1 t-\frac{\lambda+\lambda_1}{k},\frac{l}{k})\end{array}\right.\right] \tag{17}$$

provided that the conditions (13) are satisfied.

3 Conclusion

In this paper, On account of the nature of the functions in the integrand, a large number of unknown integrals can be easily obtained from it by specializing the functions and parameters involved therein. For illustration, I establish three new integrals containing the functions such as: Fox H-function, Appell polynomial, Generalised Wright Hypergeometric function, Jacobi polynomial, Generalised Riemann Zeta function, Laguerre polynomial.

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